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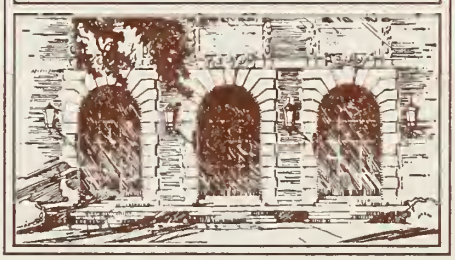

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METHODS FOR INCREASING THE COMPUTATIONAL EFFICIENCY OF INPUT-OUTPUT AND RELATED LARGE SCALE MATRIX OPERATIONS

By

Ahmed H. Sameh and Roger H. Bezdek

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This work was supported in part by the Advanced Research Projects Agency of the Department of Defense and was monitored by the U.S. Army Research Office-Durham under Contract No. DAHC04-72-C-0001.

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ABSTRACT

This paper develops numerical methods for greatly increasing the computational efficiency of input-output and related large scale matrix operations. Once the Leontief inverse matrix is computed according to the algorithm specified here, these methods permit the accurate determination of the effects on the inverse coefficients of changes in the transaction data with far fewer operations than required to compute the inverse "from scratch". These techniques are demonstrated by considering modifications down a column of the direct coefficients matrix, modifications along a row of the direct coefficients matrix, and disaggregation of the industries in the new system. The very large gains in efficiency of this method are indicated and the implications of these techniques for empirical and theoretical input-output research are briefly discussed.

ACKNOWLEDGEMENT

Ahmed Sameh is Assistant Professor in the Department of Computer Science and the Center for Advanced Computation, University of Illinois at Urbana-Champaign. Roger H. Bezdek is Assistant Professor in the Department of Economics, the Institute of Labor and Industrial Relations, and the Center for Advanced Computation. The authors are grateful to Jonathan Lermit and Daniel Slotnick for helpful comments on this paper, but retain sole responsibility for the opinions expressed here and for any errors. This work was supported in part by the Advanced Research Projects Agency of the Department of Defense under Contract No. DAHC04-72-C-0001 to the Center for Advanced Computation.

I. INTRODUCTION

When input-output analysis was first developed, the computational problems involved in interindustry economics were formidable, due to the relatively primitive state of the electronic computers then in existence. In the 1950's a literature evolved concerning various ways of handling input-output data so as to minimize the actual computations involved, but recent advances in computer technology have greatly decreased the significance of computational problems for most interindustry work. Nevertheless, the detailed BEA 1963 input-output table contained 478 industries and work with matrices of this order still can present serious computational and financial problems.[1] Further, computing costs can present a problem even with interindustry systems far more aggregate if coefficient changes necessitate the frequent recomputation of the Leontief inverse.

In this paper we develop methods for accurate calculation of the effects of changes in the original transactions matrix on the inverse coefficients which do not require the recomputation of a new inverse. While discussed in terms of input-output analysis, it is to be emphasized that the techniques developed here do not require the assumptions of input-output analysis to be valid and are thus generally applicable to any type of large-scale matrix system.

II. PRELIMINARY REMARKS

An input-output model which represents a complete economic system is referred to as a Leontief model and is a convenient way of representing an interindustry transaction table is by a partitioned Leontief matrix:

$$L = \left[\begin{array}{c:c} Z & y \\ \hdashline w^t & 0 \end{array}\right] \tag{1}$$

where y is a final demand column vector, Z is an intermediate product flow matrix, w^t is a value added row vector, and L is the Leontief matrix.[2] Letting X denote a diagonal matrix of industry outputs, we can obtain the technical coefficient matrix A by,

$$A = ZX^{-1} \tag{2}$$

The elements a_{ij} of A show the direct purchases made by the j^{th} industry from the i^{th} industry per dollar of output. Letting e and I denote, respectively, a vector of 1's and the identity matrix, we have

$$x = Ze + y = Ax + y = (I - A)^{-1} y \tag{3}$$

$(I-A)^{-1}$ is the Leontief inverse matrix and the elements of it indicate the output requirements generated directly and indirectly from industry i by industry j per dollar's worth of output delivered to final demand. The elements $a_{ij} \geq 0$, and $\sum_{i}^{n} a_{ij} \leq 1$ with the strict inequality holding for at least one column j; assuming that A is irreducible, the spectral radius of A, $\rho(A) < 1$ and $(I-A)^{-1}$ exists.[3]

In input-output analysis we are often concerned with solving the system of equations $(I-A)\,x = y$ repeatedly for various changes in the elements of A. This system of equations expresses gross output

requirements as a function of final demand and the technological structure of the economy, and changes in the elements of A can come about for a variety of reasons. The elements a_{ij} are observations subject to errors and are often revised after the original matrix has been inverted, changes along a single row or column of A can indicate technological change, and, further, it is often desirable to disaggregate a single row or column of A to obtain additional industry detail. The methods developed below permit these solutions to be obtained efficiently, accurately, and inexpensively for the first time. Similar computational procedures are discussed by Golub in statistical applications and by Saunders and Lermit in linear programming.[4]

In order to be able to apply the numerically stable algorithms developed in the following sections, the original matrix inverse must be computed in the following manner. Let us denote (I - A) by B. In order to solve the system of equations

$$Bx = y \tag{4}$$

we construct an orthogonal matrix Q ($Q^T Q = I$) such that

$$QB = R \tag{5}$$

in which R is upper triangular. One choice for Q is

$$Q = Q_{n-1} \; Q_{n-2} \; \cdots \; Q_1 \tag{6}$$

where the Q_i's are elementary Hermitian matrices (Householder's transformations)[5], each of the form

$$Q_i = I - \alpha_i u_i u_i^t \tag{7}$$

with $\alpha_i = 2/u_i^t u_i$. Let the original matrix B be denoted by B_1, then the reduction to the triangular form is determined by the recursive relation $B_{k+1} = (I - \alpha_k u_k u_k^t) B_k \quad k = 1, 2, \ldots, n-1$ (8)

where B_n is upper triangular, and

$$u_k^t = (0, \ldots, 0, b_{k,k} \pm S_k, b_{k+1,k}, \ldots, b_{n,k}) \qquad (9)$$

in which $S_k^2 = \sum_{i=k}^{n} b_{ik}^2$, the sign of S_k in (9) is chosen such that

$$|b_{k,k} \pm S_k| = |b_{k,k}| + |S_k|$$

The elements b_{ij} in the above expression are those of B_k. Therefore from (4) we have, QBx = Qy or Rx = g and x is obtained by backward substitution. If the inverse of B is desired we replace y by e_i (the i^{th} column of the identity matrix), i = 1, 2, ..., n, and the solutions x_i constitute the columns of $(I - A)^{-1}$.

In the triangularization of B, QB = R, $\frac{2}{3} n^2$ multiplications are required if Q is the product of (n-1) Householder's transformations, and $\frac{4}{3} n^3$ if Q is the product of plane rotations, (Givens reduction[5]). Usually n ranges from 100 to 400. This method of deriving the inverse coefficients (Householder's reduction), requires twice as many multiplications as Gaussian elimination. Nevertheless, it is more numerically stable and as we shall demonstrate, the savings made by never having to recompute the inverse "from scratch" make this sacrifice of computational efficiency well worthwhile.

III. COLUMN MODIFICATIONS

Each column of the direct coefficients matrix A shows the distribution of the inputs required by a specific industry, and the situation often arises where we wish to calculate the total impact of a change in this distribution. Changes in relative prices may encourage substitution of certain inputs for others, technological change can have profound influence on the makeup of an industry's inputs, or the shortage of a certain type of energy source or raw material may force an industry to alter the composition of its inputs. Obviously, these changes in the input structure will not alter the properties of the original system, and $B = (I - A)$ will still be nonsingular.

Let

$$\tilde{B} = B + [0 \vdots g_i \vdots 0]$$

then from (2) we have

$$Q\tilde{B} = R + [0 \vdots w_i \vdots 0]$$

where $w_i = Qg_i$ and $Q\tilde{B}$ will be upper triangular except for the i^{th} column. Let P be a permutation given by,

$$P = \begin{bmatrix} I_{i-1} & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & J_{n-i+1} \end{bmatrix} \tag{10}$$

in which

$$J_{n-i+1} = \begin{bmatrix} 0 & \vdots & 1 \\ \cdots\cdots & \vdots & \cdots\cdots \\ I_{n-i} & \vdots & 0 \end{bmatrix}$$

therefore

$$\tilde{QBP} = \begin{bmatrix} R_1 & \vdots & C \\ \cdots & \vdots & \cdots \\ 0 & \vdots & H \end{bmatrix} \tag{11}$$

where R_1 and H are upper triangular and upper Hessenberg matrices of order (i-1) and (n-i+1) respectively. Let Z be an orthogonal matrix such that

$$Z\tilde{QBP} = \begin{bmatrix} R_1 & \vdots & C \\ \cdots & \vdots & \cdots \\ 0 & \vdots & R_2 \end{bmatrix} = \tilde{R} \tag{12}$$

is upper triangular. An economical way of doing this is to choose Z as

$$Z = Z_{n-1} \cdots Z_{i+1} Z_i \tag{13}$$

where each Z_k is an orthogonal matrix (plane rotation) given by

$$Z_k = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c_k & s_k & & \\ & & & -s_k & c_k & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{row } k \\ \leftarrow \text{row } k+1 \\ \\ \\ \\ \end{matrix} \tag{14}$$

in which $c_k = \cos \alpha_k$, $s_k = \sin \alpha_k$, and the angles α_k are chosen so as to annihilate the element in the position (k, k+1). Let $H \equiv H^{(i)}$. Then at the k^{th} step,

$$Z_k Z_{k-1} \cdots Z_1 \tilde{QBP} = \begin{bmatrix} R_1 & \vdots & C \\ \cdots & \vdots & \cdots \\ 0 & \vdots & H^{(k+1)} \end{bmatrix} \tag{15}$$

therefore,

$$-s_k h_{kk}^{(k)} + c_k h_{k+1,k}^{(k)} = 0$$

and $$s_k = h_{k+1,k}^{(k)}/r_k, \quad c_k = h_{kk}^{(k)}/r_k \tag{16}$$

where $r_k = [h_{kk}^2 + h_{k+1,k}^2]^{\frac{1}{2}}$ in which h_{ij} belongs to $H^{(k)}$. Only rows k and k+1 of $H^{(k)}$ are affected,

$$h_{kj}^{(k+1)} = c_k h_{kj}^{(k)} + s_k h_{k+1,j}^{(k)}$$

$$h_{k+1,j}^{(k+1)} = -s_k h_{kj}^{(k)} + c_k h_{k+1,j}^{(k)} \tag{17}$$

Denoting the orthogonal matrix ZQ by $\tilde{Q}$, we have

$$\tilde{Q}\tilde{B}P = \tilde{R} \tag{18}$$

The system Bx = y can be solved as follows,

$$(\tilde{Q}\tilde{B}P)\ (P^t x) = \tilde{Q}y\ ;$$

i.e., $$\tilde{R}q = \tilde{y} \tag{19}$$

where $\tilde{y} = \tilde{Q}y$, solve for q by backward substitution then x = Pq.

Thus if $\tilde{B}$ is a matrix in which only the i^{th} column is different from that of B, then the number of multiplications required to solve the system $\tilde{B}x = f$ are roughly, $[\frac{5}{2} n^2 + 2(n-i)^2]$, i.e., a maximum of only $\frac{9}{2} n^2$ instead of $\frac{2}{3} n^3$ if we were to solve the problem "from scratch". Obviously, if i = n the problem is trivial.

IV. ROW MODIFICATIONS

In interindustry analysis we are usually more concerned with the effects of changes in the elements in a row of the direct coefficients matrix than with modifications in the column elements. The row distribution of the A matrix shows the sales of an industry's output to all industries; this information has been applied to a variety of situations. One of the more important of these is market analysis for an industry, where input-output analysis is used to determine the effects on an industry's sales of changes in economic conditions or changes in activity within other industries.[6] The analysis of the total effects of changes in the distribution of an industry's output can be accomplished in two stages:

(i) removing row b_i^t of B

Let,

$$B_1 = B - \begin{bmatrix} 0 \\ \cdots\cdots \\ b_i^t \\ \cdots\cdots \\ 0 \end{bmatrix} \tag{20}$$

then it can be easily shown that

$$B_1^t B_1 = B^t B - b_i b_i^t \tag{21}$$

from (5) we see that,

$$R^t R = B^t B$$

Since $B_1^t B_1$ is a symmetric matrix then we can write

$$R_1^t R_1 = R^t R - b_i b_i^t \tag{22}$$

Let v be a column vector defined by $R^t v = b_i$, then from (22) we obtain

$$R_1^t R_1 = R^t (I - vv^t) R$$

Now we can construct an orthogonal matrix Z such that $Zv = \alpha e_1$, thus

$$R_1^t R_1 = R^t Z^t (I - \alpha^2 e_1 e_1^t) ZR \tag{23}$$

It can be easily shown that $\mathbf{v}^t\mathbf{v} = \alpha^2 = 1$, i.e., R_1 is singular, as it should be, and

$$R_1^t R_1 = R^t Z^t D^2 Z R \tag{24}$$

in which $D^2 = \text{diag}\,(0, 1, \ldots, 1)$. Hence

$$R_1 = \tilde{P} D Z R \tag{25}$$

where $\tilde{P}$ is an orthogonal matrix that reduces the matrix (DZR) to the upper triangular form. The orthogonal matrix A can be constructed such that ZR is not a full matrix but an upper Hessenberg one. This can be done if we take

$$Z = Z_1 Z_2 \cdots Z_{n-1} \tag{26}$$

where each Z_k is as given by (14). The angles α_k are chosen as follows: Let $v \equiv v^{(n-1)}$, and define $v^{(k)}$ $(k = 1, 2, \ldots, n-1)$ by

$$v^{(k-1)} = Z_k v^{(k)} \tag{27}$$

where $v^{(k)}$ is of the form

$$v^{(k)^t} = (v_1^{(k)}, v_2^{(k)}, \ldots, v_{k+1}^{(k)}, 0, \ldots, 0)$$

From (27) we see that $v_i^{(k-1)} = v_i^{(k)}$ for $i = 1, 2, \ldots, k-1$, and

$$\begin{aligned} v_k^{(k-1)} &= c_k\, v_k^{(k)} + s_k\, v_{k+1}^{(k)} \\ v_{k+1}^{(k-1)} &= -s_k\, v_k^{(k)} + c_k\, v_{k+1}^{(k)} \end{aligned} \tag{28}$$

The angle α_k is determined such that $v_{k+1}^{(k-1)}$ is annihilated, thus

$$s_k = v_{k+1}^{(k)}/p, \quad c_k = v_k^{(k)}/p \tag{29}$$

where $p = [(v_k^{(k)})^2 + (v_{k+1}^{(k)})^2]^{\frac{1}{2}}$, and

$$v_k^{(k-1)} = p \tag{30}$$

Thus, since ZR is an upper Hessenberg matrix and D = diag (0, 1, ..., 1) it is clear that $\tilde{P}$ in (25) is a permutation matrix given by

$$\tilde{P} = \begin{bmatrix} 0 & \vdots & I_{n-1} \\ \cdots & \vdots & \cdots \\ 1 & \vdots & 0 \end{bmatrix} \tag{31}$$

and R_1 is upper triangular with the last diagonal element being zero.

(ii) adding a new row d_i^t

Let,

$$B_2 = B_1 + \begin{bmatrix} 0 \\ \cdots \\ d_i^t \\ \cdots \\ 0 \end{bmatrix} \tag{32}$$

therefore,

$$B_2^t B_2 = B_1^t B_1 + d_i d_i^t$$

Again since $B_2^t B_2$ is symmetric we can write,

$$R_2^t R_2 = R_1^t R_1 + d_i d_i^t \tag{33}$$

this equation is satisfied if $\hat{Z}$ is an orthogonal matrix of order (n+1) such that

$$\hat{Z} \begin{bmatrix} R_1 \\ \cdots \\ d_i^t \end{bmatrix} = \begin{bmatrix} R_2 \\ \cdots \\ 0 \end{bmatrix} \tag{34}$$

We choose $\hat{Z}$ as,

$$\hat{Z} = Z_1 Z_2 \cdots Z_{n-1} Z_n \tag{35}$$

where the Z_k's (k = 1, ..., n-1) are as given in (14) and Z_n is the permutation matrix $\begin{bmatrix} 0 & \vdots & 1 \\ \cdots & \vdots & \cdots \\ I_n & \vdots & 0 \end{bmatrix}$, thus the matrix

$$Z_n \begin{bmatrix} R_1 \\ \cdots\cdots \\ d_i^t \end{bmatrix} \text{ is of the form } \begin{bmatrix} \hat{H}_{(n\times n)} \\ \cdots\cdots\cdots\cdots \\ 0(1\times n) \end{bmatrix}$$

where $\hat{H}$ is an upper Hessenberg matrix. The angles α_k of Z_k ($k = 1, 2, \ldots, n-1$) are chosen such that $\hat{H}$ is reduced to an upper triangular matrix R_2.

Thus if we wish to solve the system

$$B_2 x = y \tag{37}$$

where B_2 is same as the original matrix B except of the i^{th} row, we premultiply both sides of (37) by B_2^t,

$$B_2^t B_2 x = B_2^t y$$

$$(R_2^t R_2)\, x = \hat{y} \tag{38}$$

where $\hat{y} = B_2^t y$. By forward substitution we obtain q in $R_2^t q = \hat{y}$, then by backward substitution we solve for x in $R_2 x = q$.

Changing the i^{th} row of B and solving the new system $B_2 x = y$ requires roughly $\frac{13}{2} n^2$ multiplications and $2(n-1)$ square roots, which is again a substantial saving compared to $\frac{2}{3} n^3$ multiplications.

V. DISAGGREGATION

Disaggregation of the industry classification scheme--the division of industry sectors into detailed industries--is one of the most frequently discussed topics in input-output analysis. This is important from a theoretical point of view where economists have sought to determine "optimal"levels of disaggregation for input-output tables, but it is especially significant from an empirical standpoint.[7] Individual firms and industries do not possess the resources to construct their own interindustry tables and usually have to disaggregate the rows and columns of the Commerce Department table in which they are specifically interested. Even the government often disaggregates certain industries after the original interindustry study is completed. For example, after the 1958 U. S. study had been completed the Commerce Department compiled additional transactions data on industries 14, food and kindred products, 38, primary nonferrous metals manufacturing, and 68, electric, gas, water and sanitary services, and disaggregated these three industries into 15 more detailed industries.[8] The 1958 direct coefficients matrix was thus expanded from 82 industries to 94 industries, but primarily for computational reasons a 94-order inverse matrix was never derived. The techniques developed here can be used to compute an expanded inverse from the original inverse matrix and thus avoid the effort of computing a new (I-A) inverse every time the transaction data are disaggregated.

Let us suppose that we wish to expand the i^{th} row and column of the original matrix, of order n, into m rows and columns. If $m \geq n$ a new matrix B' will have to be formed and Householder's transformations then used to reduce it to an upper triangular matrix as in (5). However,

if m << n substantial computational savings may be obtained by proceeding in the following manner.

Let B be partitioned as follows,

$$B = \left[\begin{array}{c:c} B_{11} \\ (i \times i) & B_{12} \\ \hdashline B_{21} & B_{22} \end{array}\right] \tag{39}$$

and form the (m+n × m+n) matrix C as,

$$C = \left[\begin{array}{c:c:c} B_{11} & C_1 & B_{12} \\ \hdashline C_4 & C_2 & C_5 \\ \hdashline B_{21} & \underbrace{C_3}_{m} & B_{22} \end{array}\right] \begin{matrix} \\ \\ \}m \end{matrix} \tag{40}$$

where the i^{th} row of C_1 and the i^{th} column of C_4 are zero. The rest of the elements of C_1, ..., C_5 constitute the m rows and columns to replace the i^{th} row and column of B. Thus the rank of C $\geq$ n+m-1.

Let π_1 be a permutation matrix given by,

$$\pi_1 = \begin{bmatrix} I_i & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n-i} & 0 \end{bmatrix} \tag{41}$$

Therefore $\hat{C} = \pi_1^t \, C \, \pi_1$ is given by,

$$\hat{C} = \left[\begin{array}{c:c:c} B_{11} & B_{12} & C_1 \\ \hdashline B_{21} & B_{22} & C_3 \\ \hdashline C_4 & C_5 & C_2 \end{array}\right]$$

$$= \left[\begin{array}{c:c} B & \\ \hdashline E_1 & E_2 \\ (m\times n) & \end{array}\right] \tag{42}$$

Define the $(m+n \times m+n)$ orthogonal matrix T_1 by,

$$T_1 = \left[\begin{array}{c:c} Q & 0 \\ \hdashline 0 & I_m \end{array}\right]$$

where Q is given by (2), thus

$$T_1 \left[\begin{array}{c} B \\ \hdashline E_1 \end{array}\right] = \left[\begin{array}{c} R \\ \hdashline E_1 \end{array}\right] \tag{43}$$

We can construct an orthogonal matrix T_2 as a product of several plane rotations (since $m << n$, otherwise we choose T_2 as the product of Householder's transformations) such that

$$T_2 T_1 \left[\begin{array}{c} B \\ \hdashline E_1 \end{array}\right] = \left[\begin{array}{c} R_1 \\ \hdashline 0 \end{array}\right] \tag{44}$$

Hence,

$$T_2 T_1 \hat{C} = \left[\begin{array}{c:c} R_1 & E_3 \\ \hdashline 0 & E_4 \end{array}\right] \tag{45}$$

where $\begin{bmatrix} E_3 \\ \cdots \\ E_4 \end{bmatrix} = T_2 T_1 E_2$.

Now we can choose an orthogonal matrix $\tilde{T}$ and a permutation $\tilde{\pi}$ both of order m such that $\tilde{T}E_4\tilde{\pi}$ is an upper triangular matrix R_2 of rank $\geq$ m-1. Let

$$T_3 = \left[\begin{array}{c:c} I_n & 0 \\ \hdashline 0 & \tilde{T} \end{array}\right], \text{ and } \pi_2 = \left[\begin{array}{c:c} I_n & 0 \\ \hdashline 0 & \tilde{\pi} \end{array}\right]$$

therefore,

$$(T_3T_2T_1\pi_2)\ (\pi_2^t\hat{C}\pi_2) = \left[\begin{array}{c:c} R_1 & E_3 \\ \hdashline 0 & R_2 \end{array}\right] = \hat{R} \tag{46}$$

Without loss of generality assume that $\pi_2 \equiv I$. Postmultiplying both sides by a permutation P as given in (10) except n is replaced by m+n, then

$$(T_3T_2T_1P)\ (P^t\hat{C}P) = \hat{R}P$$

where the $(m+n)^{th}$ row and column of $\tilde{B} = P^t\hat{C}P$ are the i^{th} row and column of $\hat{C}$. Now $\hat{R}P$ is of the same form as the right-hand side of (11), we can then choose an orthogonal matrix T_4 such that

$$(T_4T_3T_2T_1P)\ (P^t\hat{C}P) = \tilde{R} \tag{47}$$

is upper triangular. Writing (47) as

$$T\tilde{B} = \tilde{R} \tag{48}$$

where T is orthogonal, then if we partition $\tilde{B}$ and $\tilde{R}$ as

$$\tilde{B} = \left[\begin{array}{c:c} B^* & \\ \cdots\cdots & z \\ w^t & \\ (1\times m+n-1) & \end{array}\right], \quad \tilde{R} = \left[\begin{array}{c:c} R^* & \\ \cdots\cdots & r \\ 0 & \\ (1\times m+n-1) & \end{array}\right]$$

removing the last column we obtain

$$T\begin{bmatrix} B^* \\ \cdots\cdots \\ w^t \end{bmatrix} = \begin{bmatrix} R^* \\ \cdots\cdots \\ 0 \end{bmatrix} \tag{49}$$

Therefore,

$$B^{*t}B^* = R^{*t}R^* - ww^t \tag{50}$$

Since R^* is of rank (m+n-1), then

$$B^{*t}B^* = R^{*t}(I - \tilde{v}\tilde{v}^t)R^*$$

where $\tilde{v} = (R^*)^{-t}w$, i.e. $\tilde{v}$ is obtained by forward substitution in $R^{*t}\tilde{v} = w$. If V is an orthogonal matrix such that $V\tilde{v} = \beta e_1$ we get

$$B^{*t}B^* = U^tU = R^{*t}V^t(I - \beta^2 e_1 e_1^t)VR^* \tag{51}$$

where U is upper triangular. Since $B^{*t}B^*$ is a positive definite matrix then $\beta^2 < 1$ and

$$U^tU = R^{*t}V^t\tilde{D}^2VR^* \tag{52}$$

where $\tilde{D}^2 = (1 - \beta^2, 1, \ldots, 1)$.

Similar to section III we construct V such that VR^* is an upper Hessenberg matrix. Consequently we construct an orthogonal matrix $Z^* = Z_{N-1} \ldots Z_2 Z_1$, where N = m+n-1, (similar to (26)), such that $Z^*\tilde{D}VR^*$ is upper triangular; i.e.

$$\underset{(NxN)}{U} = Z^*\tilde{D}VR^* \tag{53}$$

Returning to our original question which was to solve the system

$$B'x = (\pi_3^t B^* \pi_3)\, x = y \tag{54}$$

where y is now of dimension (m+n-1) and π_3 is a permutation given by,

$$\pi_3 = \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & 0 & I_{n-i} \\ 0 & I_m & 0 \end{bmatrix} \tag{55}$$

we have,

$$B^* (\pi_3 x) = (\pi_3 y)$$

$$(B^{*t} B^*)\, (\pi_3 x) = B^{*t}\, \pi_3 y \quad ;$$

i.e., $$U^t U q = \tilde{y} \tag{56}$$

By forward substitution we obtain $\tilde{q}$ in $U^t \tilde{q} = \tilde{y}$, and by backsbustitution we obtain q in $Uq = \tilde{q}$. Finally, $x = \pi_3^t q$.

Note that for all the previous modifications we need only store Q and R, (QB = R). In addition, if we wish to only change rows, we need not even store Q. If it is desired to disaggregate other rows and columns these operations can be repeated for each disaggregation.

In expanding the i^{th} row and column of B into m rows and columns and solving the resulting system of equations, the operations involved are roughly $(\alpha n^2 + \beta n + \gamma)$ multiplications where $\alpha = \frac{13}{2} + 3m$, $\beta = m\,(4m + 13)$, $\gamma = m^2\,(\frac{2}{3}\, m + \frac{13}{2})$. Therefore if $m << n$ the operations are again of the order n^2.

VI. SUMMARY AND CONCLUSIONS

This paper has developed a method for determining the effects on the Leontief inverse matrix of specified changes in the coefficients of the direct requirements matrix which avoids the difficulty of recomputing the Leontief inverse. Once the original inverse matrix has been computed according to the algorithm given in Section II, additional inverse matrices corresponding to new sets of interindustry relationships can be computed accurately with savings in computation of order n, where n is the order of the input-output matrix. The use of these methods was demonstrated by considering three types of analyses very important in interindustry work: changes in the distribution of the direct coefficients down a single column, changes in the direct coefficients along a row of the matrix, and the disaggregation of economic sectors into detailed industries. These techniques can also be used to analyze the effects on the inverse matrix of other similar types of perturbations in the direct requirements data.

The method suggested here for computing the original inverse matrix requires roughly twice the time of an algorithm such as Gaussian elimination. Thus, if the inverse matrix is to be computed only once and no further analysis is anticipated, the techniques developed here will be of little use. However, if repeated inverse matrices corresponding to changes in technical coefficients are required, our methods can yield substantial savings in time, effort, and computer costs with no sacrifice in accuracy.

In relation to interindustry analysis the techniques developed here can have important effects for both empirical work and theoretical research. Empirically, these methods should be of great aid to the input-output user community. As has been indicated, one of the major

classes of users of interindustry data is the business community which relies on input-output data for a wide variety of purposes such as product planning and market analysis. The development of these computationally efficient methods implies that, once the original inverse matrix is computed by the method suggested here, individual corporations or commercial associations can conduct almost any type of meaningful simulation analysis they wish with a minimum investment of resources. Practical experiments concerning input-output tables at even 500-order or 600-order levels of detail are now entirely feasible for even relatively small businesses and organizations.

The potential advances these methods make possible for basic interindustry research are even more exciting. For four decades economists have speculated on the types of experiments it would be desirable to conduct to obtain a better understanding of the input-output model: repeated revisions of the technical coefficients to obtain different types of consistent solutions, various types of sensitivity analyses to determine the most critical sectors or coefficients, frequent incorporation of new data as they come available into an existing interindustry structure to derive more updated input-output tables, and other related types of large-scale empirical analyses.[9] Until recently the computational facilities did not exist which would permit this type of work with any realistically detailed interindustry system. Even with the development of the required computer hardware, however, the effort and costs involved in this type of large-scale matrix simulation analysis has prohibited it from being even attempted by all but the most generously funded research projects. The techniques presented here will make a substantial contribution toward reducing the complexity and costs of this type of work and thus make it

feasible to a wide range of researchers. Thus, no longer must economists be concerned with deriving "optimal" methods of aggregating input-output data sets to manageable size or with attempting to determine theoretically the total effects of different types of errors, changes, or patterns of variation in the technical coefficients.

Leontief's classic work The Structure of the American Economy was subtitled "An Empirical Application of Equilibrium Analysis". The intelligent use of the methods developed in this paper can aid greatly to our understanding of the detailed workings of an empirical equilibrium system.

Footnotes

1 The 1963 BEA input-output matrix in 367 industry detail is given in [11]; the 1963 matrix in 478 industry detail has not been published.

2 The 0 appears in the lower right hand corner of the Leontief matrix because all nonproduction accounts of the system are assumed to be consolidated.

3 A proof of this is given in Ortega [7].

4 See Golub [5], Saunders [8], and Lermit [6].

5 See Wilkinson [10].

6 For a discussion of the marketing uses of input-output data see Evans [3].

7 See Ghosh [4] and Stone [9], Chapter 8.

8 The additional 1958 transaction data for these industries are given in [1].

9 For an interesting discussion of these issues see Arnold [2].

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UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*

Center for Advanced Computation
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801

2a. REPORT SECURITY CLASSIFICATION

UNCLASSIFIED

2b. GROUP

3. REPORT TITLE

Methods for Increasing the Computational Efficiency of Input-Output and Related Large Scale Matrix Operations

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Research Report

5. AUTHOR(S) *(First name, middle initial, last name)*

Ahmed A. Sameh and Roger H. Bezdek

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
February 1973	28	11

8a. CONTRACT OR GRANT NO.

DAHC04 72-C-0001

b. PROJECT NO.

ARPA Order No. 1899

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

CAC Document No. 66

9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT

Copies may be requested from the address given in (1) above.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

U. S. Army Research Office-Durham
Duke Station, Durham, North Carolina

13. ABSTRACT

This paper develops numerical methods for greatly increasing the computational efficiency of input-output and related large scale matrix operations. Once the Leontief inverse matrix is computed according to the algorithm specified here, these methods permit the accurate determination of the effects on the inverse coefficients of changes in the transaction data with far fewer operations than required to compute the inverse "from scratch." These techniques are demonstrated by considering modifications down a column of the direct coefficients matrix, modifications along a row of the direct coefficients matrix, and disaggregation of the industries in the new system. The very large gains in efficiency of this method are indicated and the implications of these techniques for empirical and theoretical input-output research are briefly discussed.

DD FORM 1 NOV 68 1473

UNCLASSIFIED

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Input-Output Matrix						
Plane Rotations						
Householder's Transformations						
Linear Equations						

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